The Hodge Theorem:

Based on Warner's Foundations of Diff'ble Mfds and Taylor's PDE I.

Part I: The Cast

We'll be working within the class of compact, oriented, Riemannian manifolds of dimension n, with $\partial M = \emptyset$.

Recall that Riem. mfds carry a unique torsion free metric connection ∇, that induces a Laplacian on all tensor bundles, and in particular on functions $f \in C^\infty(M)$, via the equivalent expressions

$$\Delta f = \operatorname{tr}[\nabla(df)] = \operatorname{tr} \nabla^2 f = \frac{1}{g^{1/2}} \partial_i (g^{1/2} g^{ij} \partial_j \tilde{f})$$

where $g = \det g_{ij}$ in local coordinates.

There is another competiting notion of the Laplacian — the Hodge Laplacian.

From here on we let $E^k(M) := \Gamma^\infty(M, \Lambda^k(T^*M))$ be the bundle of smooth differential k-forms over M.

Def: The co-differential $\delta : E^k(M) \to E^{k-1}(M)$ is the formal adjoint of the differential $d : E^{k-1}(M) \to E^k(M)$:

$$(d\omega, \eta) = (\omega, \delta\eta) \qquad \forall \omega \in E^{k-1}(M),\ \eta \in E^k(M).$$

Here, $(-,-)$ is the standard inner product on $E^p(M)$, defined by

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle \, d\mathrm{vol}$$

where $\langle -,- \rangle$ is the Euclidean inner product on $\Lambda^p(T^*M)$ defined by, e.g., declaring $\{dx^{i_1} \wedge \cdots \wedge dx^{i_p} : 1 \le i_1 < \cdots < i_p \le n\}$ an ON base.

Recall also that the Hodge Star $* : \Lambda^k \to \Lambda^{n-k}$ is defined by, e.g.,

$$\omega \wedge * \eta = \langle \omega, \eta \rangle \, d\mathrm{vol}$$

and this extends smoothly to $* : E^k \to E^{n-k}$.

Fact / Exercise: $\delta : E^k \to E^{k-1}$ is given by $\begin{cases} \delta = (-1)^{n(k+1)+1} * d * & : k \ge 1 \\ 0 & : k = 0 \end{cases}$

Def: The Hodge Laplacian is defined by $\Delta = (d+\delta)^2 = \delta d + d\delta$.

Fact: The connection Laplacian mentioned above agrees with the Hodge Laplacian on functions, but not on higher forms in general.

This is captured by the Weitzenböck Formula $\Delta^{\nabla} - \Delta^{H} = A$, where A is a 0^{th} order linear operator the depends on the curvature.

Def: The space of harmonic p-forms on M is the space

$$H^p(M) := \{\omega \in E^p(M) : \Delta\omega = 0\} = \ker(\Delta : E^p \to E^p)$$

Part II: Properties

First of all, let's let $E := \bigoplus_{p=0}^{n} E^p(M)$, and let's declare each factor to be orthogonal.

Prop: Δ is self adjoint.

Proof:
$$\langle \Delta\omega, \eta\rangle = \langle \delta d\omega, \eta\rangle + \langle d\delta\omega, \eta\rangle$$
$$= \langle \omega, \delta d\eta\rangle + \langle \omega, d\delta\eta\rangle = \langle \omega, \Delta\eta\rangle. \quad \blacksquare$$

Prop: $\Delta\omega = 0$ iff $\delta\omega = d\omega = 0$.

Proof: Sufficiency is obvious. If $\Delta\omega = 0$, we test the equation with ω:

$$0 = \langle \Delta\omega, \omega\rangle = \langle d\delta\omega, \omega\rangle + \langle \delta d\omega, \omega\rangle = \langle \delta\omega, \delta\omega\rangle + \langle d\omega, d\omega\rangle. \quad \blacksquare$$

Cor: M connected, $f \in C^\infty(M)$, $\Delta f = 0 \to f$ is const. (Note: $\partial M = \emptyset$)

Part III: The Most Beautiful Theorem in Mathematics

Theorem: (Hodge, Weyl, Kodaira – 1930's)

For each $0 \le p \le n$, $H^p(M)$ is finite dimensional, and there is an orthogonal decomposition

$$E^p(M) = \Delta(E^p(M)) \oplus H^p(M)$$
$$= \delta d(E^p(M)) \oplus d\delta(E^p(M)) \oplus H^p(M)$$
$$= \delta(E^{p+1}(M)) \oplus d(E^{p-1}(M)) \oplus H^p(M).$$

Therefore, the Poisson Equation $\Delta\omega = \alpha$ is solvable iff α is orthogonal to $H^p(M)$.

Proof: Tasty elliptic PDE theory. I'll say something about this if I have time.

Def: The Green's operator for Δ is the map $G: E^p \to E^p$ which maps $\alpha \in E^p$ to the unique soln of $\Delta\omega = \pi(\alpha)$.

Here $\pi: E^p \to (H^p)^\perp$ is the orthogonal projection of E^p onto $(H^p)^\perp$.

Prop: (i) G is bdd and self adjoint.

(ii) G is compact.

(iii) If $[T,\Delta]=0$, then $[G,T]=0$. $\quad [A,B] = AB - BA$

Proof: (i) Δ is bdd below on $H^\perp$, hence

$$|\omega| \geq |\pi(\omega)| = |\Delta G(\omega)| \geq \frac{1}{c}|G(\omega)|$$

Moreover,

$$\langle G(\omega), \eta\rangle = \langle G(\omega), \pi(\eta)\rangle = \langle G(\omega), \Delta G(\eta)\rangle = \langle \Delta G(\omega), G(\eta)\rangle$$
$$= \langle \pi(\omega), G(\eta)\rangle = \langle \omega, G(\eta)\rangle$$

(ii) Suppose $\{\alpha_i\} \subseteq E^p$ is bdd. Then $\{G(\alpha_i)\}$ is bdd, and so is $\{\Delta G(\alpha_i)\}$:

$$|\Delta G(\alpha_i)| = |\pi(\alpha_i)| \leq |\alpha_i|.$$

By a compactness theorem for Δ, $\{G(\alpha_i)\}$ has a Cauchy subseq. ∎

(iii) We can write $G = (\Delta|_{H^\perp})^{-1} \circ \pi$.

We claim that $T(H) \subset H$, and $T(H^\perp) \subset H^\perp$. Indeed if $\omega \in H$,

$$0 = [T,\Delta](\omega) = T\Delta(\omega) - \Delta T(\omega) = -\Delta(T(\omega)) \to T(\omega) \in H.$$

If $\alpha \in H^\perp$, then $\exists \omega \in E$ s.t. $\Delta\omega = \alpha$, hence $\forall \theta \in H$,

$$\langle T\alpha, \theta\rangle = \langle [T,\Delta]\omega, \theta\rangle + \langle \Delta T\omega, \theta\rangle = \langle T\omega, \Delta\theta\rangle = 0$$
$\to T\alpha \in H^\perp$.

Thus, $[T,\pi](\omega) = 0$, and on $H^\perp$ $[T, \Delta|_{H^\perp}] = 0$.

Thus, on $H^\perp$, $[T, (\Delta|_{H^\perp})^{-1}]$, and so altogether $[T,G]=0$ by

$$TG(\omega) - GT(\omega) = T\circ(\Delta|_{H^\perp})^{-1}\circ\pi(\omega) - (\Delta|_{H^\perp})^{-1}\circ\pi\, T(\omega)$$
$$= (\Delta|_{H^\perp})^{-1}\circ\pi\, T(\omega) - T(\Delta|_{H^\perp})^{-1}\circ\pi(\omega).$$

∎

Corollary: G commutes with $*, d, \delta, \Delta$.

Part IV: Consequences for Algebraic Topology

Theorem 1: Let M be a cpt, oriented Riem. mfd without bndry. Then every dR-cohomology class has a unique harmonic representative.

Proof: let $[\alpha] \in H^{p}_{dR}(M)$. By the Hodge Theorem we can write

$$\begin{aligned}\alpha = \pi(\alpha) + \pi^{\perp}(\alpha) &= \Delta G(\alpha) + \pi^{\perp}(\alpha)\\ &= \delta d\, G(\alpha) + d\delta G(\alpha) + \pi^{\perp}(\alpha)\\ &= \delta G(d\alpha) + d\delta G(\alpha) + \pi^{\perp}(\alpha)\\ &= d\delta G(\alpha) + \pi^{\perp}(\alpha)\end{aligned}$$

Thus, $\pi^{\perp}(\alpha) \in H$ has $[\pi^{\perp}(\alpha)] = [\alpha]$.

Now, suppose $\alpha_1, \alpha_2 \in H$, $\alpha_1 = \alpha_2 + d\beta$. Then since $\Delta(\alpha_1 - \alpha_2) = 0$,

$$\langle d\beta, \alpha_1 - \alpha_2\rangle = \langle \beta, \delta(\alpha_1 - \alpha_2)\rangle = \langle \beta, 0\rangle = 0$$

So $d\beta$ and $\alpha_1 - \alpha_2$ are orthogonal. Since $d\beta + (\alpha_2 - \alpha_1) = 0$, $d\beta = \alpha_1 - \alpha_2 = 0$, and we have uniqueness for the harmonic representatives in each class.

∎

Corollary 2: The dR-cohomology groups of a cpt orientable smooth mfd without bndy are all f.d.

Theorem 3: (Poincaré Duality) M as above. Define a bilinear function

$$H^{k}_{dR} \times H^{n-k}_{dR} \longrightarrow \mathbb{R}$$

by

$$([\omega], [\eta]) \longmapsto \int_M \omega \wedge \eta .$$

Then $(-,-)$ is well defined, nondegenerate, and thus determines isomorphisms

$$H^{n-k}_{dR}(M) \approx (H^{k}_{dR}(M))^{*} .$$

Proof: To see that $(-,-)$ is well defined, suppose that $[\omega_1] = [\omega_2]$, $[\eta_1] = [\eta_2]$, $\omega_1 = \omega_2 + d\alpha$, $\eta_1 = \eta_2 + d\beta$.

$$\begin{aligned}([\omega_1], [\eta_1]) = \int \omega_1 \wedge \eta_1 &= \int (\omega_2 + d\alpha) \wedge (\eta_2 + d\beta)\\ &= ([\omega_2], [\eta_2]) + \int d\alpha \wedge \eta_2 + \omega_2 \wedge d\beta + d\alpha \wedge d\beta\\ &= ([\omega_2], [\eta_2]) + \int d(\alpha \wedge \eta_2) + d(\omega_2 \wedge \beta) + d(\alpha \wedge d\beta)\\ &= ([\omega_2], [\eta_2])\end{aligned}$$

Now suppose that $[\omega] \in H^k_{dR} \setminus \{0\}$. We seek a $[\eta] \in H^{n-k}_{dR}$ so that $([\omega],[\eta]) \neq 0$. Fix any Riem. structure on M, and assume WLOG that $\omega \in H^k_\Delta$. Then $[*,\Delta] = 0$ implies that $*\omega \in H^{n-k}_\Delta$, hence $*\omega$ is closed, and so $[*\omega] \in H^{n-k}_{dR} \setminus \{0\}$. This allows us to conclude that $(-,-)$ is nondegenerate, since

$$([\omega],[*\omega]) = \int \omega \wedge *\omega = \int |\omega|^2 \neq 0 .$$

Therefore, $(-,-)$ determines an isomorphism between H^{n-k}_{dR} and $(H^k_{dR})^*$. Explicitly, define $T : H^{n-k}_{dR} \to (H^k_{dR})^*$ by

$$T([\eta]) = (-,[\eta]) .$$

Clearly T is linear, and it is injective by non-degeneracy:

$$0 = T([\eta_1] - [\eta_2]) = (-, [\eta_1]-[\eta_2]) \longrightarrow [\eta_1] = [\eta_2] .$$

We can also define a similar linear, injective map $S : H^k_{dR} \to (H^{n-k}_{dR})^*$, so we see that

$$\dim H^{n-k}_{dR} \leq \dim (H^k_{dR})^* = \dim H^k_{dR} \leq \dim (H^{n-k}_{dR})^* = \dim H^{n-k}_{dR}$$

$\longrightarrow \dim H^{n-k}_{dR} = \dim (H^k_{dR})^*$, hence T is an isomorphism. ∎

Corollary 4: M as above & connected. Then $H^n_{dR}(M) \approx \mathbb{R}$.

Remark regarding singular (co)homology:

Theorem 5: (de Rham) For M a smooth mfd,

$$\left(H^p_{dR} \simeq H^p_{\Delta^\infty} \simeq H^p_\Delta\right) \simeq \left(H_p{}^* \simeq (H^\infty_p)^*\right)$$

Putting this together with Theorem 3 we get

Theorem 6: (Poincaré Duality for singular cohomology)

$$H^k_\Delta \simeq H^k_{dR} \simeq (H^{n-k}_{dR})^* \simeq H_{n-k}$$

∎

A Proof Sketch of the Hodge Theorem:

Part I: Solving the PDE $\Delta\omega = \alpha$.

We will proceed by studying the weak form of the equation, and then prove regularity.

To motivate this, suppose that ω is a smooth soln of $\Delta\omega = \alpha$. Let $\eta \in E^p(M)$, and consider

$$\langle \alpha, \eta \rangle = \langle \Delta\omega, \eta \rangle = \langle \omega, \Delta\eta \rangle.$$

We say that a linear functional $\ell : E^p(M) \to \mathbb{R}$ is a weak solution of $\Delta\omega = \alpha$ provided

$$\ell(\Delta\eta) = \langle \alpha, \eta \rangle \qquad \forall \eta \in E^p(M).$$

Indeed, a classical soln ω is a weak soln, by the Riesz Representation: $\ell := \langle \omega, - \rangle$:

$$\ell(\Delta\eta) = \langle \omega, \Delta\eta \rangle = \langle \Delta\omega, \eta \rangle = \langle \alpha, \eta \rangle$$

Theorem 7: Let $\alpha \in E^p(M)$, and $\ell \in (E^p)^*$ a weak solution of $\Delta\omega = \alpha$. Then $\exists\, \omega \in E^p(M)$ s.t. $\ell(\eta) = \langle \omega, \eta \rangle \ \forall \eta \in E^p(M)$. Thus, $\langle \Delta\omega, \eta \rangle = \langle \omega, \Delta\eta \rangle = \ell(\Delta\eta) = \langle \alpha, \eta \rangle \to \Delta\omega = \alpha$.

Thus, we seek a weak soln to $\Delta\omega = \alpha$, relying on the above blackboxed result to provide regularity. Actually, we'll black box one more to help along the way:

Theorem 8: Let $\{\alpha_i\} \subset E^p(M)$ with $|\alpha_i| + |\Delta\alpha_i| \le C < \infty$. Then α_i has a Cauchy subsequence.

Let's see how these results help prove the Hodge Theorem.

Finite Dimensionality of $H^p = \ker\Delta$:

If H^p were ∞-dim'l, then we could find an infinite seq. of ON α_i, violating Theorem 8.

Let then $\{\omega_1, \dots, \omega_n\}$ be an ON basis of H^p.

$E^p(M) \cong \Delta(E^p) \oplus H^p$:

For any $\alpha \in E^p$, write $\alpha = \beta + \sum \langle \alpha, \omega_i \rangle \omega_i$ where $\beta \in H^\perp$. It suffices, then, to show that $H^\perp = \Delta(E^p)$.

Step 1: $\Delta(E^p) \subset H^\perp$

Suppose $\eta \in H$, $\omega \in E^p$. Then

$$\langle \Delta\omega, \eta \rangle = \langle \omega, \Delta\eta \rangle = \langle \omega, 0 \rangle = 0$$

so indeed $\Delta(E^p) \subset H^\perp$.

Step 2: $H^\perp \subset \Delta(E^p)$

Let $\alpha \in H^\perp$, and define $\ell : \Delta(E^p) \to \mathbb{R}$ by $\ell(\Delta\eta) = \langle \alpha, \eta \rangle$. That is, we define ℓ to be a "proto weak solution" of $\Delta\omega = \alpha$. We need to extend ℓ to E^p to invoke Theorem 7.

First note that ℓ is well defined. If $\Delta\eta_1 = \Delta\eta_2$, then $\eta_1 - \eta_2 \in H$, hence $\langle \alpha, \eta_1 - \eta_2 \rangle = 0$ as $\alpha \in H^\perp$.

ℓ is also bdd on $\Delta(E^p)$. To see this we'll need:

Lemma: $\Delta : H^\perp \to E^p$ is bdd below: $\exists c > 0$ s.t. $|\alpha| \leq c|\Delta\alpha|$.

Proof: Suppose otherwise that $\exists \{\alpha_k\} \subset H^\perp$ s.t. $|\alpha_k| = 1$, $|\Delta\alpha_k| < \frac{1}{k}$. WLOG $\{\alpha_k\}$ is Cauchy by Theorem 8.

Define $\ell : E^p \to \mathbb{R}$ by $\ell(\eta) = \lim \langle \alpha_k, \eta \rangle$, which exists by the above. ℓ is bdd (with norm 1 in fact) and

$$\ell(\Delta\eta) = \lim \langle \alpha_k, \Delta\eta \rangle = \lim \langle \Delta\alpha_k, \eta \rangle = 0$$

Thus, ℓ is a weak soln of $\Delta\omega = 0$, and by Theorem 7 there is indeed some $\omega \in E^p$ s.t. $\ell(\eta) = \langle \omega, \eta \rangle$.

Then, $\langle \omega, \eta \rangle = \ell(\eta) = \lim \langle \alpha_k, \eta \rangle \Rightarrow \alpha_k \to \omega \in H^\perp$ with $|\omega| = 1$, $\Delta\omega = 0$ $\rightarrow\!\!\leftarrow$. ∎

Now, let $\eta \in E^p$, and compute

$$\begin{aligned} |\ell(\Delta\eta)| = |\ell(\Delta(\pi^\perp(\eta)))| &= |\langle \alpha, \pi^\perp(\eta) \rangle| \leq |\alpha| \, |\pi^\perp(\eta)| \\ &\leq c|\alpha| \, |\Delta(\pi^\perp(\eta))| = c|\alpha| \, |\Delta\eta| \end{aligned}$$

By Hahn-Banach, we can extend ℓ to E^p, invoke Theorem 7, and obtain an $\omega \in E^p$ s.t. $\Delta\omega = \alpha$, as desired.

Step 3: $\Delta(E^p) \simeq \delta d(E^p) \oplus d\delta(E^p)$

$$\langle \delta d\alpha, d\delta\beta \rangle = \langle d\alpha, d^2\delta\beta \rangle = 0 \rightarrow \delta d(E^p) \perp d\delta(E^p) \quad ∎$$